

Morphing Planar Graph Drawings Optimally

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Abstract. We provide an algorithm for computing a planar morph between any two planar straight-line drawings of any n -vertex plane graph in $O(n)$ morphing steps, thus improving upon the previously best known $O(n^2)$ upper bound. Further, we prove that our algorithm is optimal, that is, we show that there exist two planar straight-line drawings Γ_s and Γ_t of an n -vertex plane graph G such that any planar morph between Γ_s and Γ_t requires $\Omega(n)$ morphing steps.

1 Introduction

A *morph* is a continuous transformation between two topologically equivalent geometric objects. The study of morphs is relevant for several areas of computer science, including computer graphics, animation, and modeling. Many of the geometric shapes that are of interest in these contexts can be effectively described by two-dimensional planar graph drawings. Hence, designing algorithms and establishing bounds for morphing planar graph drawings is an important research challenge. We refer the reader to [6,7,8,11,12] for extensive descriptions of the applications of graph drawing morphs.

It has long been known that there always exists a *planar morph* (that is, a morph that preserves the planar topology of the graph at any time instant) transforming any planar straight-line drawing Γ_s of a plane graph G into any other planar straight-line drawing Γ_t of G . However, the first proof of such a result, published by Cairns in 1944 [4], was “existential”, meaning that no guarantee was provided on the complexity of the trajectories followed by the vertices during the morph. Almost 40 years later, Thomassen proved in [13] that a morph between Γ_s and Γ_t always exists in which vertices follow trajectories of exponential complexity (in the number of vertices of G). In other words, adopting a setting defined by Grünbaum and Shepard [9] which is also the one we consider in this paper, Thomassen proved that there exists a sequence $\Gamma_s = \Gamma_1, \Gamma_2, \dots, \Gamma_k = \Gamma_t$ of planar straight-line drawings of G such that, for every $1 \leq i \leq k-1$, the *linear morph* transforming Γ_i into Γ_{i+1} is planar, where a linear morph moves each vertex at constant speed along a straight-line trajectory.

A breakthrough was recently obtained by Alamdari *et al.* by proving that a planar morph between any two planar straight-line drawings of the same n -vertex connected plane graph exists in which each vertex follows a trajectory of polynomial complexity [1]. That is, Alamdari *et al.* showed an algorithm to perform the morph in $O(n^4)$

morphing steps, where a morphing step is a linear morph. The $O(n^4)$ bound was shortly afterwards improved to $O(n^2)$ by Angelini *et al.* [2].

In this paper, we provide an algorithm to compute a planar morph with $O(n)$ morphing steps between any two planar straight-line drawings Γ_s and Γ_t of any n -vertex connected plane graph G . Further, we prove that our algorithm is optimal. That is, for every n , there exist two drawings Γ_s and Γ_t of the same n -vertex plane graph (in fact a path) such that any planar morph between Γ_s and Γ_t consists of $\Omega(n)$ morphing steps. To the best of our knowledge, no super-constant lower bound was previously known.

The schema of our algorithm is the same as in [1,2]. Namely, we morph Γ_s and Γ_t into two drawings Γ_s^x and Γ_t^x in which a certain vertex v can be contracted onto a neighbor x . Such contractions generate two straight-line planar drawings Γ_s' and Γ_t' of a smaller plane graph G' . A morph between Γ_s' and Γ_t' is recursively computed and suitably modified to produce a morph between Γ_s and Γ_t . The main ingredient for our new bound is a drastically improved algorithm to morph Γ_s and Γ_t into Γ_s^x and Γ_t^x . In fact, while the task of making v contractible onto x is accomplished with $O(n)$ morphing steps in [1,2], we devise and use properties of monotone drawings, level planar drawings, and hierarchical graphs to perform it with $O(1)$ morphing steps.

The idea behind the lower bound is that linear morphs can poorly simulate rotations, that is, a morphing step rotates an edge of an angle whose size is $O(1)$. We then consider two drawings Γ_s and Γ_t of an n -vertex path P , where Γ_s lies on a straight-line, whereas Γ_t has a spiral-like shape, and we prove that in any planar morph between Γ_s and Γ_t there is one edge of P whose total rotation describes an angle whose size is $\Omega(n)$.

The rest of the paper is organized as follows. In Section 2 we give some definitions and preliminaries; in Section 3 we present our algorithm; in Section 4 we discuss the lower bound; finally, in Section 5 we conclude and offer some open problems.

2 Preliminaries

In this section we give some definitions and preliminaries.

2.1 Drawings and Embeddings

A *planar straight-line drawing* of a graph maps each vertex to a distinct point in the plane and each edge to a straight-line segment between its endpoints so that no two edges cross. A planar drawing partitions the plane into topologically connected regions, called *faces*. The bounded faces are *internal*, while the unbounded face is the *outer face*. A planar straight-line drawing is *convex* if each face is delimited by a convex polygon. A planar drawing of a graph determines a circular ordering of the edges incident to each vertex, called *rotation system*. Two drawings of a graph are *equivalent* if they have the same rotation system and the same outer face. A *plane embedding* is an equivalence class of planar drawings. A graph with a plane embedding is called a *plane graph*. A plane graph is *maximal* if no edge can be added to it while maintaining its planarity.

2.2 Subgraphs and Connectivity

A *subgraph* $G'(V', E')$ of a graph $G(V, E)$ is a graph such that $V' \subseteq V$ and $E' \subseteq E$; G' is *induced* if, for every $u, v \in V'$, $(u, v) \in E'$ if and only if $(u, v) \in E$. If G

is a plane graph, then a subgraph G' of G is regarded as a plane graph whose plane embedding is the one obtained from G by removing all the vertices and edges not in G' .

A graph G is *connected* if there is a path between every pair of vertices; it is *k-connected* if removing any $k - 1$ vertices leaves G connected; a *separating k-set* is a set of k vertices whose removal disconnects G . A *separating 3-cycle* in a plane graph G is a cycle with three vertices containing vertices both in its interior and in its exterior. Every separating 3-set in a maximal plane graph G induces a separating 3-cycle.

2.3 Monotonicity

An arc \mathbf{xy} is a line segment having x and y as endpoints and directed from x to y . An arc \mathbf{xy} is *monotone* with respect to an oriented straight line \mathbf{d} if it has a *positive projection* on \mathbf{d} . That is, let p and q be any two distinct points in this order along \mathbf{xy} when traversing \mathbf{xy} from x to y ; then, the projection of p on \mathbf{d} precedes the projection of q on \mathbf{d} when traversing \mathbf{d} according to its orientation. A path $P = (u_1, \dots, u_n)$ is *d-monotone* if the straight-line arc $\mathbf{u_i u_{i+1}}$ is monotone with respect to \mathbf{d} , for $i = 1, \dots, n - 1$; a path P is *monotone* if there exists an oriented straight line \mathbf{d} such that P is *d-monotone*. A polygon Q is *d-monotone* if there exist two vertices s and t in Q such that the two paths that start at s , that end at t , and that compose Q are both *d-monotone*. Finally, a polygon Q is *monotone* if there exists an oriented straight line \mathbf{d} such that Q is *d-monotone*. We show some lemmata about monotone paths and polygons.

Lemma 1. *Let Q be any convex polygon and let \mathbf{d} be any oriented straight line not perpendicular to any straight line through two vertices of Q . Then Q is *d-monotone*.*

Proof: Refer to Fig. 1. Denote by u_1, \dots, u_k the vertices of Q , in any order. Let \mathbf{d} be any oriented straight line not perpendicular to any straight line through two vertices of Q . For $1 \leq i \leq k$, let u'_i be the projection of u_i on \mathbf{d} . Since Q is convex and \mathbf{d} is not perpendicular to any straight line through two vertices of Q , we have that u'_i and u'_j are distinct, for $1 \leq i \neq j \leq k$. Let σ be the total order of the projections u'_i as they are encountered when traversing \mathbf{d} according to its orientation. Let u'_a and u'_b be the first and the last element in σ , respectively. We claim that the two paths P_1 and P_2 connecting u_a and u_b along Q are *d-monotone*. The claim directly implies the lemma.

We prove the claim by induction on k . If $k = 3$, then the claim is trivially proved. If $k \geq 4$, then let u'_c be the second element in σ . Assume, w.l.o.g., that u_c is in P_1 . Then, let Q' be the convex polygon obtained from Q by inserting a segment connecting u_c with the second vertex of P_2 , say u_d , and by removing u_a and its two incident segments. Let $\sigma' = \sigma \setminus \{u'_a\}$. By assumption, u'_c and u'_b are the first and the last element in σ' , respectively. By induction, the two paths $P'_1 = P_1 \setminus \{(u_a, u_c)\}$ and $P'_2 = P_2 \setminus \{(u_a, u_d)\} \cup \{u_c, u_d\}$ are *d-monotone*. Finally, arcs $\mathbf{u_a u_c}$ and $\mathbf{u_a u_d}$ have positive projections on \mathbf{d} , by the assumption that u'_a is the first element in σ . Hence, paths P_1 and P_2 are *d-monotone* and polygon Q is *d-monotone*. \square

Lemma 2. *Let $P = (u_1, u_2, u_3, u_4)$ be a path drawn in the plane. Denote by α the angle spanned by segment $\overline{u_1 u_2}$ while rotating such a segment clockwise around u_2 until it overlaps segment $\overline{u_2 u_3}$. Also, denote by β the angle spanned by segment $\overline{u_2 u_3}$*

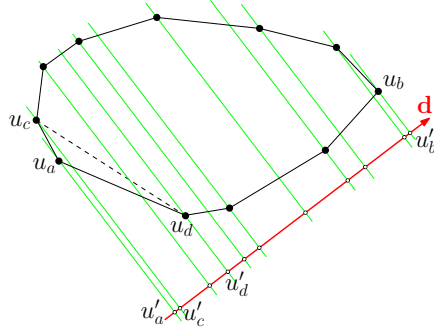


Fig. 1. Illustration for the proof of Lemma 1.

while rotating such a segment clockwise around u_3 until it overlaps segment $\overline{u_3u_4}$. Then, P is monotone if and only if $\pi < \alpha + \beta < 3\pi$.

Proof: Let $\alpha' = 2\pi - \alpha$ and $\beta' = 2\pi - \beta$ be the two angles incident to u_2 and to u_3 different from α and from β , respectively. Observe that $\pi < \alpha' + \beta' < 3\pi$ if and only if $\pi < \alpha + \beta < 3\pi$.

First, suppose that P is monotone, that is, there exists an oriented straight line d such that P is d -monotone. We prove that $\pi < \alpha + \beta < 3\pi$. Refer to Fig. 2(a). Denote by u'_1 and u'_4 the projections of u_1 and u_4 on d , respectively. Consider polygon $Q = (u_1, u_2, u_3, u_4, u'_4, u'_1)$. Denote by $\delta_1, \delta_4, \delta'_1$, and δ'_4 the angles incident to u_1, u_4, u'_1 , and u'_4 inside Q , respectively. We have $\alpha + \beta + \delta_1 + \delta_4 + \delta'_1 + \delta'_4 = 4\pi$. Further, $\delta'_1 = \delta'_4 = \pi/2$. Since $0 < \delta_1, \delta_4 < \pi$, it follows that $\pi < \alpha + \beta < 3\pi$.

Second, suppose that $\pi < \alpha + \beta < 3\pi$. We prove that P is monotone. We assume that $\alpha + \beta \leq 2\pi$. Indeed, if $\alpha + \beta > 2\pi$, then $\alpha' + \beta' \leq 2\pi$ and a symmetric proof can be exhibited in which α' and β' replace α and β . Also, assume that $\alpha \leq \beta$, as the case $\beta \leq \alpha$ can be dealt with symmetrically.

If $\alpha > \pi/2$, then $\pi/2 < \beta < 3\pi/2$. Refer to Fig. 2(b). Let d be the oriented straight line parallel to segment $\overline{u_2u_3}$ and oriented in such a way that arc u_2u_3 has a positive projection on d . Since $\alpha, \beta > \pi/2$ and since $\alpha, \beta < 3\pi/2$, it follows that arcs u_1u_2 and u_3u_4 have a positive projection on d as well, hence P is d -monotone.

If $\alpha \leq \pi/2$, then let ϵ be an arbitrarily small positive value such that $\beta > \pi - \alpha + \epsilon$. Such an ϵ always exist, given that $\beta > \pi - \alpha$. Refer to Fig. 2(c). Let l_3 be the line through u_3 such that the angle spanned by $\overline{u_2u_3}$ while clockwise rotating such a segment around u_3 until it overlaps l_3 is equal to $\pi - \alpha + \epsilon$. Let d be an oriented straight line orthogonal to l_3 and directed so that arc u_2u_3 has a positive projection on it. Observe that segment $\overline{u_2u_3}$ is not perpendicular to d , given that $\overline{u_2u_3}$ and l_3 form an angle of $\pi - \alpha + \epsilon < \pi$. We claim that P is d -monotone. Arc u_2u_3 has a positive projection on d by construction. The angle spanned by a clockwise rotation of segment $\overline{u_1u_2}$ around u_2 bringing $\overline{u_1u_2}$ to overlap with a line orthogonal to d passing through u_2 is ϵ by construction. Hence, arc u_1u_2 has a positive projection on d , given that $0 < \epsilon < \pi$. Finally, to prove that arc u_3u_4 has a positive projection on d , it suffices to observe that u_4 is in the half-

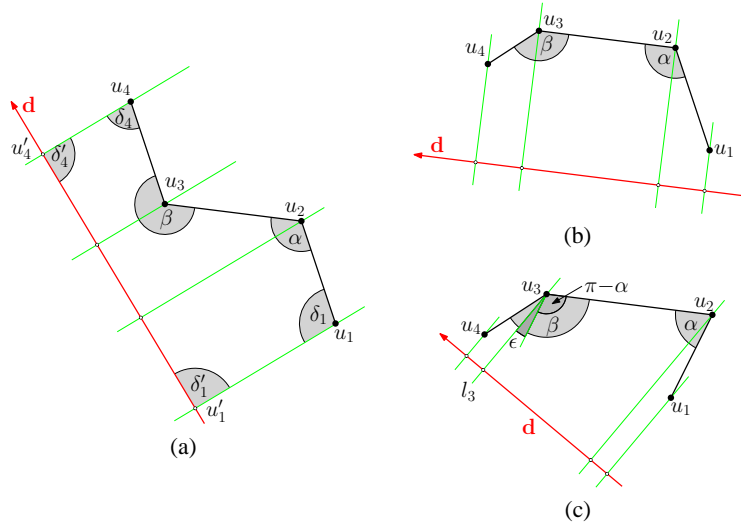


Fig. 2. (a) If P is monotone, then $\pi < \alpha + \beta < 3\pi$. (b) If $\pi < \alpha + \beta < 3\pi$ and $\alpha > \pi/2$, then P is monotone. (c) If $\pi < \alpha + \beta < 3\pi$ and $\alpha \leq \pi/2$, then P is monotone.

plane delimited by l_3 and not containing u_2 , as a consequence of $\beta > \pi - \alpha + \epsilon$ and $\alpha + \beta \leq 2\pi$.

This concludes the proof of the lemma. \square

Lemma 3. Any planar polygon Q with at most 5 vertices is monotone.

Proof: The proof distinguishes three cases, depending on the number of vertices of Q .

- If Q has three vertices, then it is convex, hence the statement follows from Lemma 1.
- If Q has four vertices, then it suffices to show that Q contains a monotone path with four vertices. Namely, assume that Q contains a path $P = (u_1, u_2, u_3, u_4)$ which is monotone with respect to some oriented straight line d . Then, paths (u_1, u_2, u_3, u_4) and (u_1, u_4) are both d -monotone, hence Q is d -monotone.
Denote by α , β , γ , and δ the angles internal to Q in clockwise order around Q . Since $\alpha + \beta + \gamma + \delta = 2\pi$, it follows that $\alpha + \beta < 3\pi$, that $\beta + \gamma < 3\pi$, that $\gamma + \delta < 3\pi$, and that $\delta + \alpha < 3\pi$. Suppose that for two consecutive angles in Q , say α and β , it holds $\alpha + \beta < \pi$; then, $\pi < \gamma + \delta < 3\pi$, and hence Q contains a monotone path with four vertices by Lemma 2. Thus, if Q does not contain any monotone path with four vertices, then every two consecutive angles in Q sum up to exactly π , hence Q is convex, and it is monotone with respect to every oriented straight line d by Lemma 1.
- If Q has five vertices, then again it suffices to show that Q contains a monotone path with four vertices. Namely, assume that Q contains a monotone path $P = (u_1, u_2, u_3, u_4)$. By definition of monotone path, there exists an oriented straight line d such that arcs u_1u_2 , u_2u_3 , and u_3u_4 have positive projections

on d . Slightly perturb the slope of d , if necessary, so that no line through two vertices of Q is orthogonal to d . If the perturbation is small enough, then P is still d -monotone. Denote by u_5 the fifth vertex of Q and, for $1 \leq i \leq 5$, denote by u'_i the projection of u_i on d . If u'_5 precedes u'_1 on d , then paths $(u_5, u_1, u_2, u_3, u_4)$ and (u_5, u_4) are both d -monotone, hence Q is d -monotone; if u'_5 follows u'_4 on d , then paths $(u_1, u_2, u_3, u_4, u_5)$ and (u_1, u_5) are both d -monotone, hence Q is d -monotone; finally, if u'_5 follows u'_1 and precedes u'_4 on d , then paths (u_1, u_2, u_3, u_4) and (u_1, u_5, u_4) are both d -monotone, hence Q is d -monotone.

Denote by $\alpha, \beta, \gamma, \delta$, and ϵ the angles internal to Q in clockwise order around Q . Since $\alpha + \beta + \gamma + \delta + \epsilon = 3\pi$, it follows that $\alpha + \beta < 3\pi$, that $\beta + \gamma < 3\pi$, that $\gamma + \delta < 3\pi$, that $\delta + \epsilon < 3\pi$, and that $\epsilon + \alpha < 3\pi$. Suppose next that $\alpha + \beta \leq \pi$, that $\beta + \gamma \leq \pi$, that $\gamma + \delta \leq \pi$, that $\delta + \epsilon \leq \pi$, and that $\epsilon + \alpha \leq \pi$. Summing up the inequalities gives $2\alpha + 2\beta + 2\gamma + 2\delta + 2\epsilon \leq 5\pi$, hence $\alpha + \beta + \gamma + \delta + \epsilon \leq 5\pi/2$, a contradiction to the fact that $\alpha + \beta + \gamma + \delta + \epsilon = 3\pi$. Hence, for at least a pair of consecutive angles of Q , say α and β , it holds $\pi < \alpha + \beta < 3\pi$. Thus, by Lemma 2, Q contains a monotone path with four vertices.

This concludes the proof of the lemma. \square

2.4 Morphing

A *linear morph* $\langle \Gamma_1, \Gamma_2 \rangle$ is a continuous transformation between two straight-line planar drawings Γ_1 and Γ_2 of a plane graph G such that each vertex moves at constant speed along a straight line from its position in Γ_1 to the one in Γ_2 . A linear morph is *planar* if no crossing or overlap occurs between any two edges or vertices during the transformation. A planar linear morph is also called a *morphing step*. In the remainder of the paper, we will construct *unidirectional* linear morphs, that were defined in [3] as linear morphs in which the straight-line trajectories of the vertices are parallel.

A *morph* $\langle \Gamma_s, \dots, \Gamma_t \rangle$ between two straight-line planar drawings Γ_s and Γ_t of a plane graph G is a finite sequence of morphing steps that transforms Γ_s into Γ_t . A *unidirectional morph* is such that each of its morphing steps is unidirectional.

Let Γ be a planar straight-line drawing of a plane graph G . The *kernel* of a vertex v of G in Γ is the open convex region R such that placing v at any point of R while maintaining unchanged the position of every other vertex of G yields a planar straight-line drawing of G . If a neighbor x of v lies on the boundary of the kernel of v in Γ , we say that v is *x -contractible*. The *contraction of v onto x* in Γ is the operation resulting in: (i) a simple graph $G' = G/(v, x)$ obtained from G by removing v and by replacing each edge (v, w) , where $w \neq x$, with an edge (x, w) (if it does not already belong to G); and (ii) a planar straight-line drawing Γ' of G' such that each vertex different from v is mapped to the same point as in Γ . Also, the *uncontraction of v from x into Γ* is the reverse operation of the contraction of v onto x in Γ , i.e., the operation that produces a planar straight-line drawing Γ of G from a planar straight-line drawing Γ' of G' .

A vertex v in a plane graph G is a *quasi-contractible vertex* if (i) $\deg(v) \leq 5$ and, (ii) for any two neighbors u and w of v , if u and w are adjacent, then (u, v, w) is a face of G . We have the following.

Lemma 4. (Angelini et al. [2]) *Every plane graph contains a quasi-contractible vertex.*

In the remainder of the paper, even when not explicitly specified, we will only consider and perform contractions of quasi-contractible vertices.

Let Γ_1 and Γ_2 be two straight-line planar drawings of the same plane graph G . We define a *pseudo-morph* of Γ_1 into Γ_2 as follows: (A) a unidirectional morph with m morphing steps of Γ_1 into Γ_2 is a pseudo-morph with m steps of Γ_1 into Γ_2 ; (B) a unidirectional morph with m_1 morphing steps of Γ_1 into a straight-line planar drawing Γ_1^x of G , followed by a pseudo-morph with m_2 steps of Γ_1^x into a straight-line planar drawing Γ_2^x of G , followed by a unidirectional morph with m_3 morphing steps of Γ_2^x into Γ_2 is a pseudo-morph of Γ_1 into Γ_2 with $m_1 + m_2 + m_3$ steps; and (C) denote by Γ_1' and Γ_2' the straight-line planar drawings of the plane graph G' obtained by contracting a quasi-contractible vertex v of G onto x in Γ_1 and in Γ_2 , respectively; then, the contraction of v onto x , followed by a pseudo-morph with x steps of Γ_1' into Γ_2' , followed by the uncontraction of v from x into Γ_2 is a pseudo-morph with $m + 2$ steps of Γ_1 into Γ_2 .

Pseudo-morphs have two useful and powerful features.

First, it is easy to design an inductive algorithm for constructing a pseudo-morph between any two planar straight-line drawings Γ_1 and Γ_2 of the same n -vertex plane graph G . Namely, consider any quasi-contractible vertex v of G and let x be any neighbor of v . Morph unidirectionally Γ_1 and Γ_2 into two planar straight-line drawings Γ_1^x and Γ_2^x , respectively, in which v is x -contractible. Now contract v onto x in Γ_1^x and in Γ_2^x obtaining two planar straight-line drawings Γ_1' and Γ_2' , respectively, of the same $(n - 1)$ -vertex plane graph G' . Then, the algorithm is completed by inductively computing a pseudo-morph of Γ_1' into Γ_2' .

Second, computing a pseudo-morph between Γ_1 and Γ_2 leads to computing a planar unidirectional morph between Γ_1 and Γ_2 , as formalized in Lemma 5. We remark that, although Lemma 5 has never been stated as below, its proof can be directly derived from the results of Alamdari et al. [1,2] and, mainly, of Barrera-Cruz et al. [3].

Lemma 5. *Let Γ_s and Γ_t be two straight-line planar drawings of a plane graph G . Let \mathcal{P} be a pseudo-morph with m steps transforming Γ_s into Γ_t . It is possible to construct a planar unidirectional morph M with m morphing steps transforming Γ_s into Γ_t .*

Proof: The proof is by induction primarily on the number k of contractions in \mathcal{P} and secondarily on the number x of steps of \mathcal{P} .

If $k = 0$, then we are in Case (A) of the definition of a pseudo-morph; hence, \mathcal{P} is a planar unidirectional morph with x morphing steps transforming Γ_s into Γ_t .

If $k > 0$ and the first step of \mathcal{P} is a unidirectional morphing step transforming Γ_s into a straight-line planar drawing Γ_s' of G , then we are in Case (B) of the definition of a pseudo-morph; denote by \mathcal{P}' the pseudo-morph composed of the last $m - 1$ steps of \mathcal{P} . By induction, there exists a planar unidirectional morph M' with $m - 1$ morphing steps transforming Γ_s' into Γ_t . Hence the first morphing step of \mathcal{P} followed by M' is a planar unidirectional morph with x morphing steps transforming Γ_s into Γ_t .

The case in which $k > 0$ and the last step of \mathcal{P} is a unidirectional morphing step can be discussed analogously.

If $k > 0$ and neither the first nor the last step of \mathcal{P} is a unidirectional morphing step, then we are in Case (C) of the definition of a pseudo-morph. Hence, the first step of \mathcal{P} is a contraction of a quasi-contractible vertex v on a neighbor x , resulting in a planar straight-line drawing Γ'_s of an $(n - 1)$ -vertex plane graph G' . Also, the last step of \mathcal{P} starts from a drawing Γ'_t of G' and uncontracts v from x into Γ_t .

Denote by \mathcal{P}' the pseudo-morph with $m - 2$ steps that is the part of \mathcal{P} transforming Γ'_s into Γ'_t . By induction, there exists a planar unidirectional morph $M' = \langle \Gamma'_s = \Gamma'_1, \dots, \Gamma'_{m-2} = \Gamma'_t \rangle$ with $m - 2$ morphing steps transforming Γ'_s into Γ'_t . For each $i = 1, \dots, x-2$, we are going to construct a drawing Γ_i of G by placing vertex v in a suitable position in Γ'_i in such a way that the morph M with m morphing steps composed of a morphing step $\langle \Gamma_s, \Gamma_1 \rangle$, followed by the morph $\langle \Gamma_1, \dots, \Gamma_{m-2} \rangle$, followed by a morphing step $\langle \Gamma_{m-2}, \Gamma_t \rangle$ is planar and unidirectional.

This strategy of constructing M starting from M' by suitably placing v in each drawing of M' is the same that was applied in [1,2,3]. It should be noted that the algorithm for placing v in $\Gamma'_1, \dots, \Gamma'_{x-2}$ differs slightly in those three papers. We opt here for an algorithm almost identical to the one in [3], as it ensures that M is a unidirectional morph. However, since in [3] G is assumed to be a maximal plane graph, vertex v can always be chosen to be an internal vertex of G with degree at least 3. In our case, instead, v might be incident to the outer face of G and might have degree 1 or 2.

We now describe the algorithm in [3] for placing v when v is internal and $\deg(v) = 5$; then, we will argue that an analogous technique can be applied even if v is incident to the outer face of G and has degree 1 or 2.

Observe that, at any time instant t during M' , there exists a disk of radius $\epsilon_t > 0$ that is centered at m and that does not contain any vertex or edge other than x and its incident edges. Let $\epsilon = \min_t \{\epsilon_t\}$ be the minimum ϵ_t among all time instants t of M' .

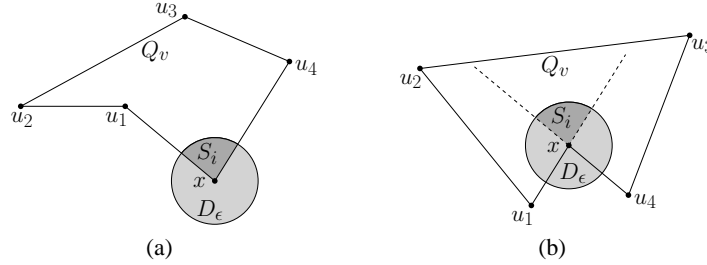


Fig. 3. Circular sector S_i if (a) the internal angle of Q_v incident to x is smaller than or equal to π or (b) the internal angle of Q_v incident to x is larger than π .

For each $i = 1, \dots, m-2$, let S_i be the circular sector resulting from the intersection between a disk D_ϵ centered at x with radius ϵ and the kernel of the polygon Q_v induced by the neighbors of v in Γ_i . In particular (see Fig. 3), if the internal angle of Q_v incident to x is smaller than or equal to π , then S_i is delimited by the two radii of D_ϵ that overlap with the two edges of Q_v incident to x , while if such an angle is larger than π then S_i is delimited by the two radii of D_ϵ that overlap with the elongations emanating from x

of the two edges of Q_v incident to x . Barrera-Cruz et al. prove in [3] that each circular sector S_i contains at least one *nice* point, defined as follows. All the points of S_{m-2} are nice. For $i = 1, \dots, m-3$, a point p_i of S_i is nice if there exists a nice point p_{i+1} in S_{i+1} such that the line passing through p_i and p_{i+1} is parallel to the trajectory followed by each vertex during the unidirectional morphing step transforming Γ'_i into Γ'_{i+1} . The proof in [3] is completed by showing that placing v on the nice point p_i in Γ'_i and on the corresponding nice point p_{i+1} in Γ'_{i+1} yields two drawings Γ_i and Γ_{i+1} of G such that $\langle \Gamma_i, \Gamma_{i+1} \rangle$ is planar and, by construction, unidirectional.

In order to adapt this algorithm to our setting, it is sufficient to describe how to compute each circular sector S_i , since the rest of the proof works exactly as described in [3] for the case in which $\deg(v) = 5$. The complication here is that the neighbors of v might not create a polygon Q_v enclosing v in its interior, hence it is not possible to use the concept of “kernel of a polygon” in order to define S_i . To overcome this problem, we use the concept of “kernel of a vertex” v , defined as the region of the plane such that each of its points has direct visibility to all the neighbors of v . Observe that this is the same property satisfied by the kernel of Q_v , however the kernel of v is well-defined even if the neighbors of v do not induce a polygon enclosing v in its interior, e.g., if v is incident to the outer face or $\deg(v) \leq 2$.

More in detail, if $\deg(v) = 1$, then S_i is the intersection of D_ϵ with the region of Γ'_i representing the face of G' that contains v in G . If $\deg(v) = 2$, then S_i is the intersection of: (i) D_ϵ , (ii) the region of Γ'_i representing the face of G' that contains v in G , and (iii) the half-plane that is to the left (right) of the oriented straight line from a neighbor w of v to the other neighbor z of v if w, z , and v appear in this counter-clockwise (resp. clockwise) order along cycle (w, z, v) in G . Finally, if $3 \leq \deg(v) \leq 5$, then let w and z be the two neighbors of x in G such that edges (x, w) , (x, v) , and (x, z) appear consecutively around x in this clockwise order; then, if the angle spanned when rotating (x, w) clockwise till coinciding with (x, z) is smaller than or equal to π , then S_i is delimited by the two radii of D_ϵ that overlap with edges (x, w) and (x, z) , otherwise S_i is delimited by the two radii of D_ϵ that overlap with the elongations of edges (x, w) and (x, z) emanating from x . We observe that an analogous definition of circular sectors S_i was provided in [2] (although the morphs constructed in [2] are not unidirectional).

We conclude the proof by observing that the first and the last morphing steps $\langle \Gamma_s, \Gamma_1 \rangle$ and $\langle \Gamma_{m-2}, \Gamma_t \rangle$ of M are planar, since v has been placed on a nice point in Γ_1 and in Γ_{m-2} , and unidirectional, since v is the only vertex moving during these two steps. \square

2.5 Hierarchical Graphs and Level Planarity

A *hierarchical graph* is a tuple $(G, \mathbf{d}, L, \gamma)$ where: (i) G is a graph; (ii) \mathbf{d} is an oriented straight line in the plane; (iii) L is a set of parallel lines (sometimes called *layers*) that are orthogonal to \mathbf{d} ; the lines in L are assumed to be ordered in the same order as they are intersected by \mathbf{d} when traversing such a line according to its orientation; and (iv) γ is a function that maps each vertex of G to a line in L in such a way that, if an edge (u, v) belongs to G , then $\gamma(u) \neq \gamma(v)$. A *level drawing* of $(G, \mathbf{d}, L, \gamma)$ (sometimes also called *hierarchical drawing*) maps each vertex v of G to a point on the line $\gamma(v)$ and each edge (u, v) of G such that line $\gamma(u)$ precedes line $\gamma(v)$ in L

to an arc uv monotone with respect to d . A *hierarchical plane graph* is a hierarchical graph (G, d, L, γ) such that G is a plane graph and such that a level planar drawing Γ of (G, d, L, γ) exists that “respects” the embedding of G (that is, the rotation system and the outer face of G in Γ are the same as in the plane embedding of G). Given a hierarchical plane graph (G, d, L, γ) , an *st-face* of G is a face delimited by two paths $(s = u_1, u_2, \dots, u_k = t)$ and $(s = v_1, v_2, \dots, v_l = t)$ such that $\gamma(u_i)$ precedes $\gamma(u_{i+1})$ in L , for every $1 \leq i \leq k - 1$, and such that $\gamma(v_i)$ precedes $\gamma(v_{i+1})$ in L , for every $1 \leq i \leq l - 1$. We say that (G, d, L, γ) is a *hierarchical plane st-graph* if every face of G is an st-face. Let Γ be any straight-line level planar drawing of a hierarchical plane graph (G, d, L, γ) and let f be a face of G ; then, it is easy to argue that f is an st-face if and only if the polygon delimiting f in Γ is d -monotone.

In this paper we will use a result of Hong and Nagamochi on the existence of convex straight-line level planar drawings of hierarchical plane st-graphs [10]. Here we explicitly formulate a weaker version of their main theorem.¹

Theorem 1. (*Hong and Nagamochi [10]*) *Let (G, d, L, γ) be a triconnected hierarchical plane st-graph. There exists a convex straight-line level planar drawing of (G, d, L, γ) .*

Let Γ be a straight-line level planar drawing of a hierarchical plane graph (G, d, L, γ) . Since each edge (u, v) of G is represented in Γ by a d -monotone arc, the fact that (u, v) intersects a line $l_i \in L$ does not depend on the actual drawing Γ , but only on the fact that l_i lies between lines $\gamma(u)$ and $\gamma(v)$ in L . Assume that each line $l_i \in L$ is oriented so that d cuts l_i from the right to the left of l_i . We say that an edge e *precedes (follows)* a vertex v on a line l_i in Γ if $\gamma(v) = l_i$, e intersects l_i in a point $p_i(e)$, and $p_i(e)$ precedes (resp. follows) v on l_i when traversing such a line according to its orientation. Also, we say that an edge e *precedes (follows)* an edge e' on a line l_i in Γ if e and e' both intersect l_i at points $p_i(e)$ and $p_i(e')$, and $p_i(e)$ precedes (resp. follows) $p_i(e')$ on l_i when traversing such a line according to its orientation.

Now consider two straight-line level planar drawings Γ_1 and Γ_2 of a hierarchical plane graph (G, d, L, γ) . We say that Γ_1 and Γ_2 are *left-to-right equivalent* if, for any line $l_i \in L$, for any vertex or edge x of G , and for any vertex or edge y of G , we have that x precedes (follows) y on l_i in Γ_1 if and only if x precedes (resp. follows) y on l_i in Γ_2 . We are going to make use of the following lemma.

Lemma 6. *Let Γ_1 and Γ_2 be two left-to-right equivalent straight-line level planar drawings of the same hierarchical plane graph (G, d, L, γ) . Then the linear morph $\langle \Gamma_1, \Gamma_2 \rangle$ transforming Γ_1 into Γ_2 is planar and unidirectional.*

In order to prove Lemma 6, we first recall an auxiliary lemma appeared in [3] stating that if two points x and y move at constant speed on the same line l and x precedes

¹ We make some remarks. First, the main result in [10] proves that a convex straight-line level planar drawing of (G, d, L, γ) exists even if a convex polygon representing the cycle delimiting the outer face of G is arbitrarily prescribed. Second, the result holds for a super-class of the triconnected planar graphs, namely for all the graphs that admit a convex straight-line drawing [5, 14]. Third, the result assumes that the lines in L are horizontal; however, a suitable rotation of the coordinate axes shows how that assumption is not necessary. Fourth, looking at the figures in [10] one might get the impression that the lines in L need to be equidistant; however, this is nowhere used in their proof, hence the result holds for any set of parallel lines.

(follows) y on l both at the beginning and at the end of the movement, then x precedes (follows) y on l during the whole movement.

Lemma 7. (Barrera-Cruz et al. [3]) *Let l be an oriented straight line and let x_0, x_1, y_0 , and y_1 be points on l . Assume that x_i precedes y_i on l , for $i = 0, 1$. Consider a point x that moves in one unit of time from x_0 to x_1 , and a point y that moves in one unit of time from y_0 to y_1 . Then, x precedes y on l during the entire movement.*

We now exhibit a proof of Lemma 6.

Proof of Lemma 6: Morph $\langle \Gamma_1, \Gamma_2 \rangle$ is clearly unidirectional. We prove that it is planar.

Lemma 7 and the fact that Γ_1 and Γ_2 are left-to-right equivalent directly imply that, if two vertices lie on the same line $l \in L$, then they never overlap during $\langle \Gamma_1, \Gamma_2 \rangle$.

We prove that there exists no overlap between a vertex u and an edge e of G during $\langle \Gamma_1, \Gamma_2 \rangle$. Such a proof also implies that there is no crossing between two edges at any time t during $\langle \Gamma_1, \Gamma_2 \rangle$; in fact, such a crossing can only happen if an end-vertex of one of the two edges overlaps the other edge at a time instant $t' \leq t$.

In order to prove that there exists no overlap between u and e , it suffices to prove that the point $p_i(e)$ in which e intersects line $l_i = \gamma(u)$ moves at constant speed during $\langle \Gamma_1, \Gamma_2 \rangle$, since in this case Lemma 7 and the fact that Γ_1 and Γ_2 are left-to-right equivalent imply that u and $p_i(e)$ never overlap.

The fact that $p_i(e)$ moves at constant speed during $\langle \Gamma_1, \Gamma_2 \rangle$ directly follows from: (i) the two end-vertices v and w of e move at constant speed on two lines $\gamma(v)$ and $\gamma(w)$ that are parallel to l_i ; and (ii) for any time instant t of $\langle \Gamma_1, \Gamma_2 \rangle$, the coefficients that express $p_i(e)$ as a convex combination of the positions of v and w are the same.

This concludes the proof of the lemma. \square

3 A Morphing Algorithm

In this section we describe an algorithm to construct a planar unidirectional morph with $O(n)$ steps between any two straight-line planar drawings Γ_s and Γ_t of the same n -vertex plane graph G . The algorithm relies on two subroutines, called FAST CONVEXIFIER and CONTRACTIBILITY CREATOR, which are described in Sections 3.1 and 3.2, respectively. The algorithm is described in Section 3.3.

3.1 Fast Convexifier

Consider a straight-line planar drawing Γ of an n -vertex maximal plane graph G , for some $n \geq 4$. Let v be a quasi-contractible internal vertex of G and let C_v be the cycle of G induced by the neighbors of v . See Fig. 4(a). In this section we show an algorithm, that we call FAST CONVEXIFIER, morphing Γ into a straight-line planar drawing Γ_M of G in which C_v is convex. Algorithm FAST CONVEXIFIER consists of a single unidirectional morphing step.

Let G' be the $(n - 1)$ -vertex plane graph obtained by removing v and its incident edges from G . Also, let Γ' be the straight-line planar drawing of G' obtained by removing v and its incident edges from Γ . We have the following lemma.

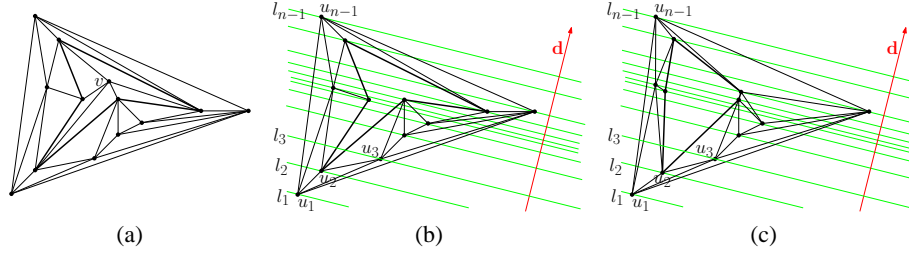


Fig. 4. (a) Straight-line planar drawing Γ of G . (b) Straight-line level planar drawing Γ' of $(G', \mathbf{d}, L', \gamma')$. (c) Convex straight-line level planar drawing Γ'_M of $(G', \mathbf{d}, L', \gamma')$.

Lemma 8. *Graph G' is 3-connected.*

Proof: Suppose, for a contradiction, that G' contains a set S' of vertices with $|S'| \leq 2$ whose removal disconnects G' . It follows that removing the vertices in $S = S' \cup \{v\}$ from G disconnects G . If $|S| = 1$ or $|S| = 2$, then G contains a separation 1-set or 2-set, respectively, in both cases contradicting the fact that G is a maximal plane graph. If $|S| = 3$, then S is a separating 3-set. However, any separating 3-set in a maximal plane graph induces a separating 3-cycle C . Hence, C contains at least one neighbor of v in its interior and at least one neighbor of v in its exterior. This contradicts the assumption that v is a quasi-contractible vertex of G . \square

Consider the polygon Q_v representing C_v in Γ and in Γ' . By Lemma 3, Q_v is \mathbf{d} -monotone, for some oriented straight line \mathbf{d} . Slightly perturb the slope of \mathbf{d} so that no line through two vertices of G in Γ is perpendicular to \mathbf{d} . If the perturbation is small enough, then Q_v is still \mathbf{d} -monotone. Denote by u_1, \dots, u_{n-1} the vertices of G' ordered according to their projection on \mathbf{d} . For $1 \leq i \leq n-1$, denote by l_i the line through u_i orthogonal to \mathbf{d} . Let $L' = \{l_1, \dots, l_{n-1}\}$; note that the lines in L' are parallel and distinct. Let γ' be the function that maps u_i to l_i , for $1 \leq i \leq n-1$. See Fig. 4(b).

Lemma 9. *$(G', \mathbf{d}, L', \gamma')$ is a hierarchical plane st-graph.*

Proof: By construction, Γ' is a straight-line level planar drawing of $(G', \mathbf{d}, L', \gamma')$, hence $(G', \mathbf{d}, L', \gamma')$ is a hierarchical plane graph. Further, every polygon delimiting a face of G' in Γ' is \mathbf{d} -monotone. This is true for Q_v by construction and for every other polygon Q_i delimiting a face of G' in Γ' by Lemma 1, given that Q_i is a triangle and hence it is convex. Since every polygon delimiting a face of G' in Γ' is \mathbf{d} -monotone, every face of G' is an st-face, hence $(G', \mathbf{d}, L', \gamma')$ is a hierarchical plane st-graph. \square

By Lemmata 8 and 9, $(G', \mathbf{d}, L', \gamma')$ is a triconnected hierarchical plane st-graph. By Theorem 1, a convex straight-line level planar drawing Γ'_M of $(G', \mathbf{d}, L', \gamma')$ exists. Denote by Q_v^M the convex polygon representing C_v in Γ'_M . See Fig. 4(c).

Denote by r and s the minimum and the maximum index such that u_r and u_s belong to C_v , respectively. Denote by $l(v)$ the line through v orthogonal to \mathbf{d} in Γ . If $l(v)$ were contained in the half-plane delimited by l_r and not containing l_s , then v would not lie

inside Q_v in Γ , as the projection of every vertex of Q_v on d would follow the projection of v on d . Analogously, $l(v)$ is not contained in the half-plane delimited by l_s and not containing l_r . It follows that $l(v)$ is “in-between” l_r and l_s , that is, $l(v)$ lies in the strip defined by l_r and l_s .

Construct a straight-line planar drawing Γ_M of G from Γ'_M by placing v on any point at the intersection of $l(v)$ and the interior of Q_v^M . Observe that such an intersection is always non-empty, given that l_r and l_s have non-empty intersection with Q_v^M , given that $l(v)$ is in-between l_r and l_s , and given that Q_v^M is a convex polygon.

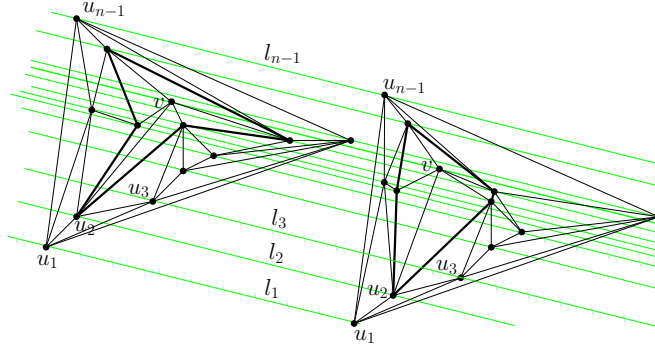


Fig. 5. Morphing Γ into a straight-line planar drawing Γ_M of G in which the polygon Q_v^M representing C_v is convex. The thick green line is $l(v)$.

Let γ be the function that maps v to $l(v)$ and u_i to l_i , for $1 \leq i \leq n-1$. We have that Γ and Γ_M are left-to-right equivalent straight-line level planar drawings of $(G, d, L' \cup \{l(v)\}, \gamma)$. By Lemma 6, the linear morph transforming Γ into Γ_M is planar and unidirectional. Further, the polygon Q_v^M representing C_v in Γ_M is convex. Thus, algorithm FAST CONVEXIFIER consists of a single unidirectional morphing step transforming Γ into Γ_M . See Fig. 5.

3.2 Contractibility Creator

In this section we describe an algorithm, called CONTRACTIBILITY CREATOR, that receives a straight-line planar drawing Γ of a plane graph G , a quasi-contractible vertex v of G , and a neighbor x of v , and returns a planar unidirectional morph with $O(1)$ morphing steps transforming Γ into a straight-line planar drawing Γ' of G in which v is x -contractible.

Denote by u_1, \dots, u_k the clockwise order of the neighbors of v . If $k = 1$, then v is x -contractible in Γ , hence algorithm CONTRACTIBILITY CREATOR returns $\Gamma' = \Gamma$.

If $k \geq 2$, consider any pair of consecutive neighbors of v , say u_i and u_{i+1} (where $u_{k+1} = u_1$). See Fig. 6(a). If edge (u_i, u_{i+1}) belongs to G , then cycle (u_i, v, u_{i+1}) delimits a face of G , given that v is quasi-contractible. Otherwise, we aim at morphing

Γ into a straight-line planar drawing of G where a dummy edge (u_i, u_{i+1}) can be introduced while maintaining planarity and while ensuring that cycle (u_i, v, u_{i+1}) delimits a face of the augmented graph $G \cup \{(u_i, u_{i+1})\}$. This is accomplished as follows:

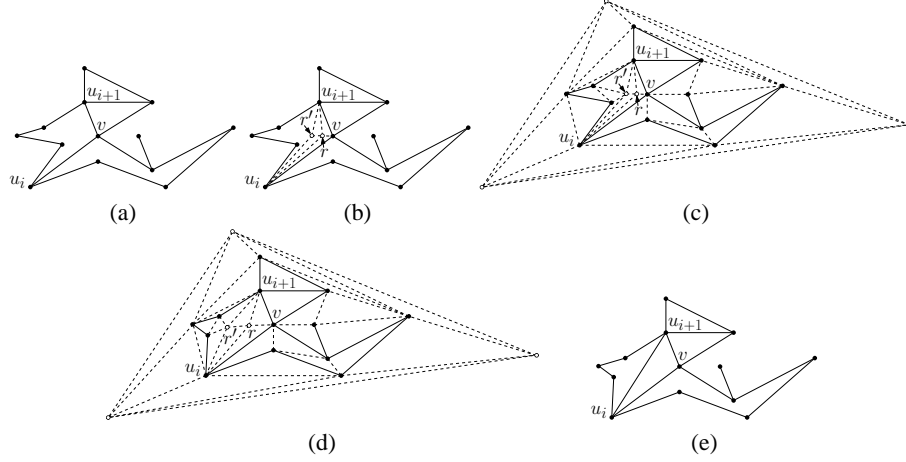


Fig. 6. (a) Drawing Γ of G . (b) Drawing Γ^+ of G^+ . (c) Drawing Γ^* of G^* . (d) Drawing Γ_M^* of G^* . (e) Drawing Γ_M of $G \cup \{(u_i, u_{i+1})\}$.

1. We add two dummy vertices r and r' , and six dummy edges (r, v) , (r, u_i) , (r, u_{i+1}) , (r', u_i) , (r', u_{i+1}) , and (r, r') to Γ and G , obtaining a straight-line planar drawing Γ^+ of a plane graph G^+ , in such a way that Γ^+ is planar and cycles (v, r, u_i) , (v, r, u_{i+1}) , (r', r, u_i) , and (r', r, u_{i+1}) delimit faces of G^+ . See Fig. 6(b).
2. We add dummy vertices and edges to Γ^+ and G^+ , obtaining a straight-line planar drawing Γ^* of a graph G^* , in such a way that Γ^* is planar, that G^* is a maximal planar graph, and that edges (u_i, u_{i+1}) and (r', v) do not belong to G^* . Observe that r is a quasi-contractible vertex of G^* . See Fig. 6(c).
3. We apply algorithm FAST CONVEXIFIER to morph Γ^* with one unidirectional morphing step into a straight-line planar drawing Γ_M^* of G^* such that the polygon of the neighbors of r is convex. See Fig. 6(d).
4. We remove from Γ_M^* all the dummy vertices and edges that belong to G^* and do not belong to G , and we add edge (u_i, u_{i+1}) to Γ_M^* and G , obtaining a straight-line planar drawing Γ_M of graph $G \cup \{(u_i, u_{i+1})\}$. See Fig. 6(e).

If $k = 2$, then after the above described algorithm is performed, we have that v is x -contractible in $\Gamma' = \Gamma_M$, both if $x = u_1$ or if $x = u_2$, given that (v, u_1, u_2) delimits a face of $G \cup \{(u_1, u_2)\}$. If $3 \leq k \leq 5$, then the above described algorithm is repeated at most k times (namely once for each pair of consecutive neighbors of v that are not adjacent in G), at each time inserting an edge between a distinct pair of

consecutive neighbors of v . Eventually, we obtain a straight-line planar drawing Φ of plane graph $G \cup \{(u_1, u_2), (u_2, u_3), (u_3, u_4), (u_4, u_5), (u_5, u_1)\}$ in which v is quasi-contractible. Then we add dummy vertices and edges to Φ , obtaining a straight-line planar drawing Σ of a graph H , in such a way that H is a maximal planar graph and that v is quasi-contractible in Σ . We apply algorithm FAST CONVEXIFIER to morph Σ with one unidirectional morphing step into a straight-line planar drawing Ψ of H such that the polygon of the neighbors of v is convex. Hence, v is contractible onto any of its neighbors in Ψ . Then, we remove the edges of H not in G , obtaining a straight-line planar drawing Γ' of G in which v is contractible onto any of its neighbors; hence, v is x -contractible in Γ' . Finally, observe that Γ' is obtained from Γ in at most $k + 1 \leq 6$ unidirectional morphing steps.

3.3 The Algorithm

We now describe an algorithm to construct a pseudo-morph \mathcal{P} with $O(n)$ steps between any two straight-line planar drawings Γ_s and Γ_t of the same n -vertex plane graph G .

The algorithm works by induction on n . If $n = 1$, then \mathcal{P} consists of a single unidirectional morphing step transforming Γ_s into Γ_t . If $n \geq 2$, then let v be a quasi-contractible vertex of G , which exists by Lemma 4, and let x be any neighbor of v . Let M_s and M_t be the planar unidirectional morphs with $O(1)$ morphing steps produced by algorithm CONTRACTIBILITY CREATOR transforming Γ_s and Γ_t into straight-line planar drawings Γ_s^x and Γ_t^x of G , respectively, such that v is x -contractible both in Γ_s^x and in Γ_t^x . Let G' be the $(n - 1)$ -vertex plane graph obtained by contracting v onto x in G , and let Γ'_s and Γ'_t be the straight-line planar drawings of G' obtained from Γ_s^x and Γ_t^x , respectively, by contracting v onto x . Further, let \mathcal{P}' be the inductively constructed pseudo-morph between Γ'_s and Γ'_t . Then, pseudo-morph \mathcal{P} is defined as the unidirectional morph M_s transforming Γ_s into Γ_s^x , followed by the contraction of v onto x in Γ_s^x , followed by the pseudo-morph \mathcal{P}' between Γ'_s and Γ'_t , followed by the uncontraction of v from x into Γ_t^x , followed by the unidirectional morph M_t^{-1} transforming Γ_t^x into Γ_t . Observe that \mathcal{P} has a number of steps which is a constant plus the number of steps of \mathcal{P}' . Hence, \mathcal{P} consists of $O(n)$ steps.

A unidirectional planar morph M between Γ_s and Γ_t can be constructed with a number of morphing steps equal to the number of steps of \mathcal{P} , by Lemma 5. This proves the following:

Theorem 2. *Let Γ_s and Γ_t be any two straight-line planar drawings of the same n -vertex plane graph G . There exists an algorithm to construct a planar unidirectional morph with $O(n)$ morphing steps transforming Γ_s into Γ_t .*

4 A Lower Bound

In this section we show two straight-line planar drawings Γ_s and Γ_t of an n -vertex path $P = (v_1, \dots, v_n)$, and we prove that any planar morph M between Γ_s and Γ_t requires $\Omega(n)$ morphing steps. In order to simplify the description, we consider each edge $e_i = (v_i, v_{i+1})$ as oriented from v_i to v_{i+1} , for $i = 1, \dots, n - 1$.

Drawing Γ_s (see Fig. 7(a)) is such that all the vertices of P lie on a horizontal straight-line with v_i to the left of v_{i+1} , for each $i = 1, \dots, n-1$.

Drawing Γ_t (see Fig. 7(b)) is such that:

- for each $i = 1, \dots, n-1$ with $i \bmod 3 \equiv 1$, the (green) segment representing e_i is horizontal with v_i to the left of v_{i+1} ;
- for each $i = 1, \dots, n-1$ with $i \bmod 3 \equiv 2$, the (blue) segment representing e_i is parallel to line $y = \tan(\frac{2\pi}{3})x$ with v_i to the right of v_{i+1} ; and
- for each $i = 1, \dots, n-1$ with $i \bmod 3 \equiv 0$, the (red) segment representing e_i is parallel to line $y = \tan(-\frac{2\pi}{3})x$ with v_i to the right of v_{i+1} .

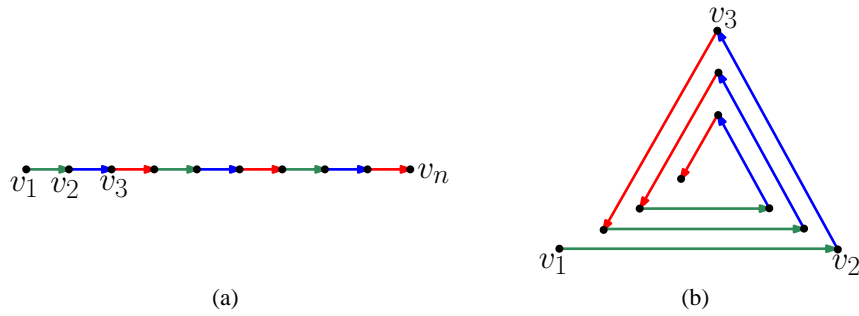


Fig. 7. Drawings Γ_s (a) and Γ_t (b).

Let $M = \langle \Gamma_s = \Gamma_1, \dots, \Gamma_m = \Gamma_t \rangle$ be any planar morph transforming Γ_s into Γ_t .

For $i = 1, \dots, n$ and $j = 1, \dots, m$, we denote by v_i^j the point where vertex v_i is placed in Γ_j ; also, for $i = 1, \dots, n-1$ and $j = 1, \dots, m$ we denote by e_i^j the directed straight-line segment representing edge e_i in Γ_j .

For $1 \leq j \leq m-1$, we define the *rotation* ρ_i^j of e_i around v_i during the morphing step $\langle \Gamma_j, \Gamma_{j+1} \rangle$ as follows (see Fig. 8). Translate e_i at any time instant of $\langle \Gamma_j, \Gamma_{j+1} \rangle$ so that v_i stays fixed at a point a during the entire morphing step. After this translation, the morph between e_i^j and e_i^{j+1} is a rotation of e_i around a (where e_i might vary its length during $\langle \Gamma_j, \Gamma_{j+1} \rangle$) spanning an angle ρ_i^j , where we assume $\rho_i^j > 0$ if the rotation is counter-clockwise, and $\rho_i^j < 0$ if the rotation is clockwise. We have the following.

Lemma 10. *For each $j = 1, \dots, m-1$ and $i = 1, \dots, n-1$, we have $|\rho_i^j| < \pi$.*

Proof: Assume, for a contradiction, that $|\rho_i^j| \geq \pi$, for some $1 \leq j \leq m-1$ and $1 \leq i \leq n-1$. Also assume, w.l.o.g., that the morphing step $\langle \Gamma_j, \Gamma_{j+1} \rangle$ happens between time instants $t = 0$ and $t = 1$. For any $0 \leq t \leq 1$, denote by $v_i(t)$, $v_{i+1}(t)$, $e_i(t)$, and $\rho_i^j(t)$ the position of v_i , the position of v_{i+1} , the drawing of e_i , and the rotation of e_i around v_i at time instant t , respectively. Note that $v_i(0) = v_i^j$, $v_{i+1}(0) = v_{i+1}^j$, $e_i(0) = e_i^j$, $\rho_i^j(0) = 0$, and $\rho_i^j(1) = \rho_i^j$. Since a morph is a continuous transformation and since $|\rho_i^j| \geq \pi$, there exists a time instant t_π with $0 < t_\pi \leq 1$ such that $|\rho_i^j(t_\pi)| = \pi$.

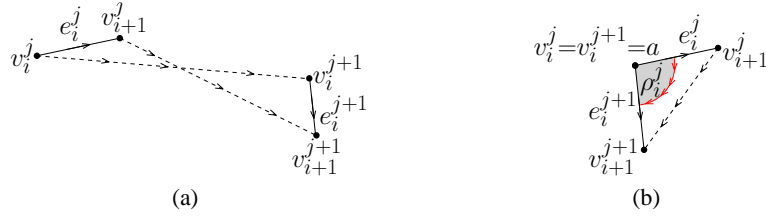


Fig. 8. Rotation ρ_i^j . (a) Morph between e_i^j and e_i^{j+1} . (b) Translation of the positions of e_i during $\langle \Gamma_j, \Gamma_{j+1} \rangle$, resulting in e_i spanning an angle ρ_i^j around v_i .

We prove that there exists a time instant t_r with $0 < t_r \leq t_\pi$ in which $v_i(t)$ and $v_{i+1}(t)$ coincide, thus contradicting the assumption that morph $\langle \Gamma_j, \Gamma_{j+1} \rangle$ is planar.

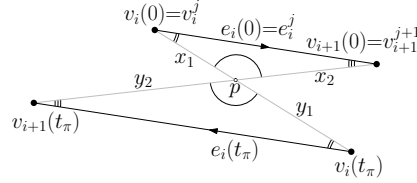


Fig. 9. Illustration for the proof of Lemma 10.

Since $|\rho_i^j(t_\pi)| = \pi$, it follows that $e_i(t_\pi)$ is parallel to $e_i(0)$ and oriented in the opposite way. This easily leads to conclude that t_r exists if $e_i(t_\pi)$ and $e_i(0)$ are aligned. Otherwise, the straight-line segments $v_i(0)v_i(t_\pi)$ and $v_{i+1}(0)v_{i+1}(t_\pi)$ meet in a point p . Refer to Fig. 9. Let $x_1 = |pv_i(0)|$, $x_2 = |pv_{i+1}(0)|$, $y_1 = |pv_i(t_\pi)|$, and $y_2 = |pv_{i+1}(t_\pi)|$. By the similarity of triangles $(v_i(0), p, v_{i+1}(0))$ and $(v_i(t_\pi), p, v_{i+1}(t_\pi))$, we have $\frac{x_1}{y_1} = \frac{x_2}{y_2}$ and hence $\frac{x_1}{x_1+y_1} = \frac{x_2}{x_2+y_2}$. Thus, $v_i(\frac{x_1}{x_1+y_1}t_\pi)$ and $v_{i+1}(\frac{x_1}{x_1+y_1}t_\pi)$ are coincident with p . This contradiction proves the lemma. \square

For $j = 1, \dots, m-1$, we denote by M_j the subsequence $\langle \Gamma_1, \dots, \Gamma_{j+1} \rangle$ of M ; also, for $i = 1, \dots, n-1$, we define the *total rotation* $\rho_i(M_j)$ of edge e_i around v_i during morph M_j as $\rho_i(M_j) = \sum_{m=1}^j \rho_i^m$.

We will show in Lemma 12 that there exists an edge e_i , for some $1 \leq i \leq n-1$, whose total rotation $\rho_i(M_{m-1}) = \rho_i(M)$ is $\Omega(n)$. In order to do that, we first analyze the relationship between the total rotation of two consecutive edges of P .

Lemma 11. *For each $j = 1, \dots, m-1$ and for each $i = 1, \dots, n-2$, we have that $|\rho_{i+1}(M_j) - \rho_i(M_j)| < \pi$.*

Proof: Suppose, for a contradiction, that $|\rho_{i+1}(M_j) - \rho_i(M_j)| \geq \pi$ for some $1 \leq j \leq m-1$ and $1 \leq i \leq n-2$. Assume that j is minimal under this hypothesis. Since each vertex moves continuously during M_j , there exists an intermediate drawing Γ^* of

P , occurring during morphing step $\langle \Gamma_j, \Gamma_{j+1} \rangle$, such that $|\rho_{i+1}(M^*) - \rho_i(M^*)| = \pi$, where $M^* = \langle \Gamma_1, \dots, \Gamma_j, \Gamma^* \rangle$ is the morph obtained by concatenating M_{j-1} with the morphing step transforming Γ_j into Γ^* . Recall that in Γ_1 edges e_i and e_{i+1} lie on the same straight line and have the same orientation. Then, since $|\rho_{i+1}(M^*) - \rho_i(M^*)| = \pi$, in Γ^* edges e_i and e_{i+1} are parallel and have opposite orientations. Also, since edges e_i and e_{i+1} share vertex v_{i+1} , they lie on the same line. This implies that such edges overlap, contradicting the hypothesis that M^* , M_j , and M are planar. \square

We are now ready to prove the key lemma for the lower bound.

Lemma 12. *There exists an index i such that $|\rho_i(M)| \in \Omega(n)$.*

Proof: Refer to Fig. 7. For every $1 \leq i \leq n-2$, edges e_i and e_{i+1} form an angle of π radians in Γ_s , while they form an angle of $\frac{\pi}{3}$ radians in Γ_t . Hence, $\rho_{i+1}(M) = \rho_i(M) + \frac{2\pi}{3} + 2z_i\pi$, for some $z_i \in \mathbb{Z}$.

In order to prove the lemma, it suffices to prove that $z_i = 0$, for every $i = 1, \dots, n-2$. Namely, in this case $\rho_{i+1}(M) = \rho_i(M) + \frac{2\pi}{3}$ for every $1 \leq i \leq n-2$, and hence $\rho_{n-1}(M) = \rho_1(M) + \frac{2\pi}{3}(n-2)$. This implies $|\rho_{n-1}(M) - \rho_1(M)| \in \Omega(n)$, and thus $|\rho_1(M)| \in \Omega(n)$ or $|\rho_{n-1}(M)| \in \Omega(n)$.

Assume, for a contradiction, that $z_i \neq 0$, for some $1 \leq i \leq n-2$. If $z_i > 0$, then $\rho_{i+1}(M) \geq \rho_i(M) + \frac{8\pi}{3}$; further, if $z_i < 0$, then $\rho_{i+1}(M) \leq \rho_i(M) - \frac{4\pi}{3}$. Since each of these inequalities contradicts Lemma 11, the lemma follows. \square

We are now ready to state the main theorem of this section.

Theorem 3. *There exists two straight-line planar drawings Γ_s and Γ_t of an n -vertex path P such that any planar morph between Γ_s and Γ_t requires $\Omega(n)$ morphing steps.*

Proof: The two drawings Γ_s and Γ_t of path $P = (v_1, \dots, v_n)$ are those illustrated in Fig. 7. By Lemma 12, there exists an edge e_i of P , for some $1 \leq i \leq n-1$, such that $|\sum_{j=1}^{x-1} \rho_i^j| \in \Omega(n)$. Since, by Lemma 10, we have that $|\rho_i^j| < \pi$ for each $j = 1, \dots, x-1$, it follows that $x \in \Omega(n)$. This concludes the proof of the theorem. \square

5 Conclusions

In this paper we presented an algorithm to construct a planar morph between two planar straight-line drawings of the same n -vertex plane graph in $O(n)$ morphing steps. We also proved that this bound is tight (note that our lower bound holds for any morphing algorithm in which the vertex trajectories are polynomial functions of constant degree).

In our opinion, the main challenge in this research area is the one of designing algorithms to construct planar morphs between straight-line planar drawings with good resolution and within polynomial area (or to prove that no such algorithm exists). In fact, the algorithm we presented, as well as other algorithms known at the state of the art [1,2,4,13], construct intermediate drawings in which the ratio between the lengths of the longest and of the shortest edge is exponential. Guaranteeing good resolution and small area seems to be vital for making a morphing algorithm of practical utility.

Finally, we would like to mention an original problem that generalizes the one we solved in this paper and that we repute very interesting. Let Γ_s and Γ_t be two straight-line drawings of the same (possibly non-planar) topological graph G . Does a morphing algorithm exist that morphs Γ_s into Γ_t and that preserves the topology of the drawing at any time instant? A solution to this problem is not known even if we allow the trajectories followed by the vertices to be of arbitrary complexity.

References

1. S. Alamdari, P. Angelini, T. M. Chan, G. Di Battista, F. Frati, A. Lubiw, M. Patrignani, V. Roselli, S. Singla, and B. T. Wilkinson. Morphing planar graph drawings with a polynomial number of steps. In S. Khanna, editor, *24th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA '13)*, pages 1656–1667. SIAM, 2013.
2. P. Angelini, F. Frati, M. Patrignani, and V. Roselli. Morphing planar graph drawings efficiently. In S. Wismath and A. Wolff, editors, *21st International Symposium on Graph Drawing (GD '13)*, volume 8242 of *LNCS*, pages 49–60. Springer, 2013.
3. F. Barrera-Cruz, P. Haxell, and A. Lubiw. Morphing planar graph drawings with unidirectional moves. Mexican Conference on Discr. Math. and Comput. Geom., 2013.
4. S. S. Cairns. Deformations of plane rectilinear complexes. *American Math. Monthly*, 51:247–252, 1944.
5. N. Chiba, T. Yamanouchi, and T. Nishizeki. Linear algorithms for convex drawings of planar graphs. In J. A. Bondy and U. S. R. Murty, editors, *Progress in Graph Theory*, pages 153–173. Academic Press, New York, NY, 1984.
6. C. Erten, S. G. Kobourov, and C. Pitta. Intersection-free morphing of planar graphs. In *11th Symposium on Graph Drawing*, pages 320–331, 2003.
7. C. Friedrich and P. Eades. Graph drawing in motion. *J. Graph Algorithms Appl.*, 6(3):353–370, 2002.
8. C. Gotsman and V. Surazhsky. Guaranteed intersection-free polygon morphing. *Computers & Graphics*, 25(1):67–75, 2001.
9. B. Grünbaum and G.C. Shephard. *The geometry of planar graphs*. Cambridge University Press, 1981.
10. S. H. Hong and H. Nagamochi. Convex drawings of hierarchical planar graphs and clustered planar graphs. *J. Discrete Algorithms*, 8(3):282–295, 2010.
11. V. Surazhsky and C. Gotsman. Controllable morphing of compatible planar triangulations. *ACM Trans. Graph*, 20(4):203–231, 2001.
12. V. Surazhsky and C. Gotsman. Intrinsic morphing of compatible triangulations. *Internat. J. of Shape Model.*, 9:191–201, 2003.
13. C. Thomassen. Deformations of plane graphs. *Journal of Combinatorial Theory, Series B*, 34(3):244–257, 1983.
14. C. Thomassen. Plane representations of graphs. In J. A. Bondy and U. S. R. Murty, editors, *Progress in Graph Theory*, pages 43–69. Academic Press, New York, NY, 1984.